32$^{\text{th}}$ BALKAN MATHEMATICAL OLYMPIAD
Athens, Hellas (May 5, 2015)

English version

Problem 1. Let a, b and c be positive real numbers. Prove that

$$a^3b^6 + b^3c^6 + c^3a^6 + 3a^3b^3c^3 \geq abc(a^3b^3 + b^3c^3 + c^3a^3) + a^2b^2c^2(a^3 + b^3 + c^3).$$

Problem 2. Let ABC be a scalene triangle with incentre I and circumcircle (ω). The lines AI, BI, CI intersect (ω) for the second time at the points D, E, F, respectively. The lines through I parallel to the sides BC, AC, AB intersect the lines EF, DF, DE at the points K, L, M, respectively. Prove that the points K, L, M are collinear.

Problem 3. A jury of 3366 film critics are judging the Oscars. Each critic makes a single vote for his favourite actor, and a single vote for his favourite actress. It turns out that for every integer $n \in \{1, 2, \dots, 100\}$ there is an actor or actress who has been voted for exactly n times. Show that there are two critics who voted for the same actor and for the same actress.

Problem 4. Prove that among any 20 consecutive positive integers there exists an integer d such that for each positive integer n we have the inequality

$$n\sqrt{d}\,\{n\sqrt{d}\} > \frac{5}{2}$$

where $\{x\}$ denotes the fractional part of the real number x. The fractional part of a real number x is x minus the greatest integer less than or equal to x.

Time allowed: 4 **hours and** 30 **minutes.**
Each Problem is worth 10 **points.**