22nd Junior Balkan Mathematical Olympiad
Rhodes 19-24 June 2018

Solutions

Problem 1. Find all the pairs (m, n) of integers which satisfy the equation

$$m^5 - n^5 = 16mn.$$

Solution. If one of m, n is 0, the other has to be 0 too, and $(m, n) = (0, 0)$ is one solution. If $mn \neq 0$, let $d = \gcd(m, n)$ and we write $m = da$, $n = db$, $a, b \in \mathbb{Z}$ with $(a, b) = 1$. Then, the given equation is transformed into

$$d^3a^5 - d^3b^5 = 16ab \tag{1}$$

So, by the above equation, we conclude that $a \mid d^3b^5$ and thus $a \mid d^3$. Similarly $b \mid d^3$. Since $(a, b) = 1$, we get that $ab \mid d^3$, so we can write $d^3 = abr$ with $r \in \mathbb{Z}$. Then, equation (1) becomes

$$abra^5 - abrb^5 = 16ab \Rightarrow$$
$$r(a^5 - b^5) = 16.$$

Therefore, the difference $a^5 - b^5$ must divide 16. Therefore, the difference $a^5 - b^5$ must divide 16. This means that

$$a^5 - b^5 = \pm 1, \pm 2, \pm 4, \pm 8, \pm 16.$$

The smaller values of $|a^5 - b^5|$ are 1 or 2. Indeed, if $|a^5 - b^5| = 1$ then $a = \pm 1$ and $b = 0$ or $a = 0$ and $b = \pm 1$, a contradiction. If $|a^5 - b^5| = 2$, then $a = 1$ and $b = -1$ or $a = -1$ and $b = 1$. Then $r = -8$, and $d^3 = -8$ or $d = -2$. Therefore, $(m, n) = (-2, 2)$. If $|a^5 - b^5| > 2$ then, without loss of generality, let $a > b$ and $a \geq 2$. Putting $a = x + 1$ with $x \geq 1$, we have

$$\begin{aligned} |a^5 - b^5| &= |(x+1)^5 - b^5| \\ &\geq |(x+1)^5 - x^5| \\ &= |5x^4 + 10x^3 + 10x^2 + 5x + 1| \geq 31 \end{aligned}$$

which is impossible. Thus, the only solutions are $(m, n) = (0, 0)$ or $(-2, 2)$.

Problem 2. Let n three-digit numbers satisfy the following properties:

(1) No number contains the digit 0.

(2) The sum of the digits of each number is 9.

(3) The units digits of any two numbers are different.

(4) The tens digits of any two numbers are different.

(5) The hundreds digits of any two numbers are different.

Find the largest possible value of n.

Solution. Let S denote the set of three-digit numbers that have digit sum equal to 9 and no digit equal to 0. We will first find the cardinality of S. We start from the number 111 and each element of S can be obtained from 111 by a string of 6 A's (which means that we add 1 to the current digit) and 2 G's (which means go to the next digit). Then for example 324 can be obtained from 111 by the string $AAGAGAAA$. There are in total

$$\frac{8!}{6! \cdot 2!} = 28$$

such words, so S contains 28 numbers. Now, from the conditions $(3), (4), (5)$, if $\overline{abc}$ is in T then each of the other numbers of the form $\overline{**c}$ cannot be in T, neither $\overline{*b*}$ can be, nor $\overline{a**}$. Since there are $a+b-2$ numbers of the first category, $a+c-2$ from the second and $b+c-2$ from the third one. In these three categories there are

$$(a+b-2)+(b+c-2)+(c+a-2) = 2(a+b+c)-6 = 2\cdot 9-6 = 12$$

distinct numbers that cannot be in T if $\overline{abc}$ is in T. So, if T has n numbers, then $12n$ are the forbidden ones that are in S, but each number from S can be a forbidden number no more than three times, once for each of its digits, so

$$n + \frac{12n}{3} \le 28 \iff n \le \frac{28}{5},$$

and since n is an integer, we get $n \le 5$. A possible example for $n = 5$ is

$$T = \{144, 252, 315, 423, 531\}.$$

□

Comment by PSC. It is classical to compute the cardinality of S and this can be done in many ways. In general, the number of solutions of the equation

$$x_1 + x_2 + \cdots + x_k = n$$

in positive integers, where the order of x_i matters, is well known that equals to $\binom{n-1}{k-1}$. In our case, we want to count the number of positive solutions to $a+b+c = 9$. By the above, this equals to $\binom{9-1}{3-1} = 28$. Using the general result above, we can also find that there are $a+b-2$ numbers of the form $\overline{**c}$.

Problem 3. Let $k > 1$ be a positive integer and $n > 2018$ be an odd positive integer. The nonzero rational numbers $x_1, x_2, \ldots, x_n$ are not all equal and satisfy

$$x_1 + \frac{k}{x_2} = x_2 + \frac{k}{x_3} = x_3 + \frac{k}{x_4} = \cdots = x_{n-1} + \frac{k}{x_n} = x_n + \frac{k}{x_1}\cdot$$

Find:

a) the product $x_1 x_2 \ldots x_n$ as a function of k and n

b) the least value of k, such that there exist $n, x_1, x_2, \ldots, x_n$ satisfying the given conditions.

a) If $x_i = x_{i+1}$ for some i (assuming $x_{n+1} = x_1$), then by the given identity all x_i will be equal, a contradiction. Thus $x_1 \neq x_2$ and

$$x_1 - x_2 = k\frac{x_2 - x_3}{x_2 x_3}.$$

Analogously

$$x_1 - x_2 = k\frac{x_2 - x_3}{x_2 x_3} = k^2 \frac{x_3 - x_4}{(x_2 x_3)(x_3 x_4)} = \cdots = k^n \frac{x_1 - x_2}{(x_2 x_3)(x_3 x_4)\ldots(x_1 x_2)}\cdot$$

Since $x_1 \neq x_2$ we get

$$x_1 x_2 \ldots x_n = \pm\sqrt{k^n} = \pm k^{\frac{n-1}{2}}\sqrt{k}.$$

If one among these two values, positive or negative, is obtained, then the other one will be also obtained by changing the sign of all x_i since n is odd.

b) From the above result, as n is odd, we conclude that k is a perfect square, so $k \geq 4$. For $k = 4$ let $n = 2019$ and $x_{3j} = 4$, $x_{3j-1} = 1$, $x_{3j-2} = -2$ for $j = 1, 2, \ldots, 673$. So the required least value is $k = 4$.

□

Comment by PSC. There are many ways to construct the example when $k = 4$ and $n = 2019$. Since $3 \mid 2019$, the idea is to find three numbers x_1, x_2, x_3 satisfying the given equations, not all equal, and repeat them as values for the rest of the x_i's. So, we want to find x_1, x_2, x_3 such that

$$x_1 + \frac{4}{x_2} = x_2 + \frac{4}{x_3} = x_3 + \frac{4}{x_1}\cdot$$

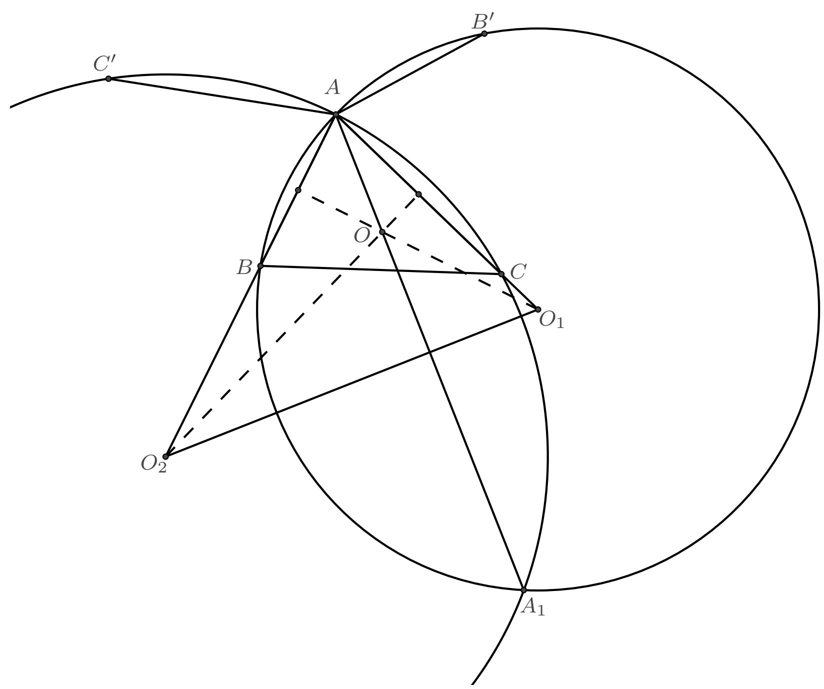
As above, $x_1 x_2 x_3 = \pm 8$. Suppose without loss of generality that $x_1 x_2 x_3 = -8$. Then, solving the above system we see that if $x_1 \neq 2$, then

$$x_2 = -\frac{4}{x_1 - 2} \quad \text{and} \quad x_3 = 2 - \frac{4}{x_1},$$

leading to infinitely many solutions. The example in the official solution is obtained by choosing $x_1 = -2$.

Problem 4. Let ABC be an acute triangle, A', B' and C' be the reflections of the vertices A, B and C with respect to BC, CA, and AB, respectively, and let the circumcircles of triangles ABB' and ACC' meet again at A_1. Points B_1 and C_1 are defined similarly. Prove that the lines AA_1, BB_1 and CC_1 have a common point.

Solution. Let O_1, O_2 and O be the circumcenters of triangles ABB', ACC' and ABC respectively. As AB is the perpendicular bisector of the line segment CC', O_2 is the intersection of the perpendicular bisector of AC with AB. Similarly, O_1 is the intersection of the perpendicular bisector of AB with AC. It follows that O is the orthocenter of triangle AO_1O_2. This means that AO is perpendicular to O_1O_2. On the other hand, the segment AA_1 is the common chord of the two circles, thus it is perpendicular to O_1O_2. As a result, AA_1 passes through O. Similarly, BB_1 and CC_1 pass through O, so the three lines are concurrent at O. □



Comment by PSC. We present here a different approach.
We first prove that A_1, B and C' are collinear. Indeed, since $\angle BAB' = \angle CAC' = 2\angle BAC$, then from the circles (ABB'), (ACC') we get

$$\angle AA_1B = \frac{\angle BA_1B'}{2} = \frac{180^\circ - \angle BAB'}{2} = 90^\circ - \angle BAC = \angle AA_1C'.$$

It follows that

$$\angle A_1AC = \angle A_1C'C = \angle BC'C = 90^\circ - \angle ABC \tag{1}$$

On the other hand, if O is the circumcenter of ABC, then

$$\angle OAC = 90^\circ - \angle ABC. \tag{2}$$

From (1) and (2) we conclude that A_1, A and O are collinear. Similarly, BB_1 and CC_1 pass through O, so the three lines are concurrent in O.